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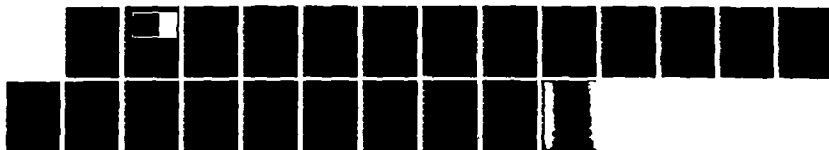
CONVERGENCE OF BIVARIATE CARDINAL INTERPOLATION(U)
WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER
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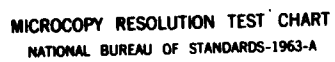
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CONVERGENCE OF BIVARIATE CARDINAL
INTERPOLATION

Carl de Boor, Klaus Höllig and
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UNIVERSITY OF WISCONSIN-MADISON
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CONVERGENCE OF BIVARIATE CARDINAL INTERPOLATION

Carl de Boor¹, Klaus Höllig¹ and Sherman Riemenschneider²

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ABSTRACT

We give necessary and sufficient conditions for the convergence of cardinal interpolation with bivariate box splines as the degree tends to infinity.

AMS (MOS) Subject Classifications: 41A05, 41A15, 41A63

Key Words: bivariate, box-splines, interpolation, convergence, exponential type, Whittaker operator

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

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SIGNIFICANCE AND EXPLANATION

This is a follow-up on the MRC TSR #2485 in which we introduced and studied interpolation by a linear combination of translates of a bivariate box spline on a three-direction mesh. This is of interest because these box splines are not just tensor products of univariate B-splines but are genuinely bivariate, yet are true generalizations of the univariate cardinal B-spline. This allows one to be guided by Schoenberg's highly successful analysis of univariate cardinal splines, while at the same time struggling with a more complicated setup.

The specific task of the present report is the derivation of necessary and of sufficient conditions for the convergence of the box spline interpolants as the degree goes to infinity. The conditions are stated in terms of the Fourier transform of the interpoland.



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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

1. Introduction and statement of main results. For a set of vectors

$\Xi = \{\xi_1, \dots, \xi_n\}$ with $\xi_v \in \mathbb{Z}^m$, the box spline M_Ξ is the functional on

$C^0(\mathbb{R}^m)$ defined by

$$(1) \quad \langle M_\Xi, \phi \rangle := \int_{[-1/2, 1/2]^n} \phi \left(\sum_{v=1}^n \lambda_v \xi_v \right) d\lambda.$$

As becomes apparent from its Fourier transform

$$(2) \quad \hat{M}_\Xi(y) = \prod_{v=1}^n \frac{\sin(\xi_v \cdot y/2)}{\xi_v \cdot y/2},$$

the box spline is a natural generalization of the univariate cardinal

spline.

Motivated by I. J. Schoenberg's beautiful results [7-9], we were led

to study cardinal interpolation for box splines. The first question is

whether or not the interpolation problem is correct; i.e. whether there

exists, for any continuous bounded function f , a unique bounded spline

$$I_\Xi f \in S_\Xi := \text{span}\{M_\Xi(\cdot - j) : j \in \mathbb{Z}^m\}$$

which interpolates f at the lattice points,

$$I_\Xi f(k) = f(k), \quad k \in \mathbb{Z}^m.$$

Clearly a necessary condition is that the translates of the box spline M_Ξ

be linearly independent. We conjecture that this is also sufficient or,

equivalently, that

$$(3) \quad P_\Xi(x) := \sum_{j \in \mathbb{Z}^m} M_\Xi(j) e^{ij \cdot x} > 0$$

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to infinity.
and the convergence of I_n depends on just how the components of n go more complicated. There is a continuum of different fundamental domains in the bivariate analogue of the above Theorem. However, the situation is One might expect that $(-\pi, \pi)^2$ plays the role of the interval $(-\pi, \pi)$ where n_v is the multiplicity of the corresponding vector in Ξ .

from now on that Ξ is of this form and refer to it by $n = (n_1, n_2, n_3) \in \mathbb{Z}_+^3$ vectors in Ξ are chosen from the set $\{(1,0), (0,1), (1,1)\}$. We assume Up to symmetry bivariate cardinal interpolation is correct iff the

$\text{supp } f \subseteq [-\pi, \pi]$.
 m converges uniformly to a bounded function f on \mathbb{R} as $m \rightarrow \infty$, then
(11) If a sequence of (univariate) cardinal splines s_m of degree

$$\|f - I_m^m f\|_{\infty, [-a, a]} \rightarrow 0, \quad m \rightarrow \infty.$$

of degree m converge locally uniformly to f , i.e. for any $a > 0$,
 $\text{supp } f \subset (-\pi, \pi)$, then the (univariate) cardinal spline interpolants $I_m^m f$
(1) If the Fourier transform of f is a tempered distribution with
Theorem [7,9].

which is due to F.B. Richards, I.J. Schoenberg and S.D. Klemenschnelder.
to infinity. We obtain the bivariate analogue of the following result
convergence of bivariate cardinal spline interpolation as the degree tends
In this paper we continue our investigation in [3] concerning the
case [3, Theorem 4].

S_Ξ . So far, the positivity of P has been proved only in the bivariate
holds if and only if the box splines $M_\Xi^{(-j)}$, $j \in \mathbb{Z}_m^+$, form a basis for

Denote by n' the "middle" component of n , i.e., the second number in any ordering of n_1, n_2, n_3 . We write

$$n \rightarrow N$$

if a sequence $n(m)$, $m \in \mathbb{N}$, satisfies

$$(n1) \quad n'(m) \rightarrow \infty \text{ as } m \rightarrow \infty,$$

$$(n2) \quad \lim_{m \rightarrow \infty} \frac{n(m)}{n'(m)} = N \in [0, \infty]^3.$$

Further, we assume that

$$(n3) \quad |n| := n_1 + n_2 + n_3 \leq c(n')^c$$

where c is some positive constant. Examples of admissible sequences are

$$n(m) = (m, 2m, 3m) \text{ with } N = (1/2, 1, 3/2),$$

$$n(m) = (1, m, m^2) \text{ with } N = (0, 1, \infty).$$

The assumption (n3) excludes degenerate cases like $n(m) = (1, m, m!)$.

The role of the interval $(-\pi, \pi)$ is played by certain domains Ω_N corresponding to the limit of the sequence n . For $N \in [0, \infty]^3$ they are defined by

$$(4) \quad \Omega_N := \{2\pi x: 0 \leq a_{N,j}(x) < 1 \text{ for } j \in J\}$$

where $J := \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ and for $x = (u,v)$, $j = (k,l)$,

$$(5) \quad a_{N,j}(x) := \left(\frac{u}{u+k}\right)^{N_1} \left(\frac{v}{v+l}\right)^{N_2} \left(\frac{u+v}{u+v+k+l}\right)^{N_3}.$$

Clearly, the set Ω_N is bounded by the curves $\Gamma_{N,j} := \{2\pi x: a_{N,j}(x) = 1\}$,

$j \in J$. If one of the components of N equals ∞ the sets Ω_N as well as the curves $\Gamma_{N,j}$ have to be interpreted as the appropriate limits (cf. Proposition 2). A qualitatively correct picture of Ω_N is given in Figure 1. Figure 2 shows a few special cases. Of particular interest is the symmetric case $N = (1,1,1)$.

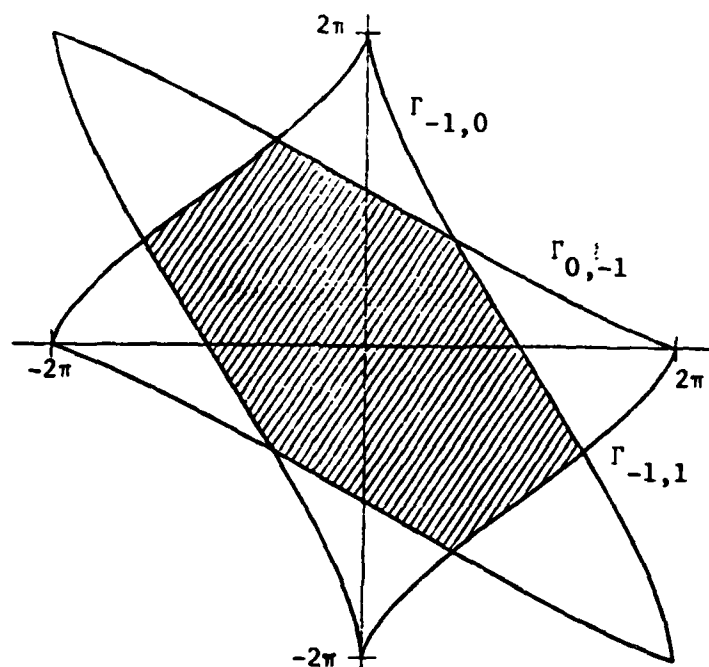


Figure 1

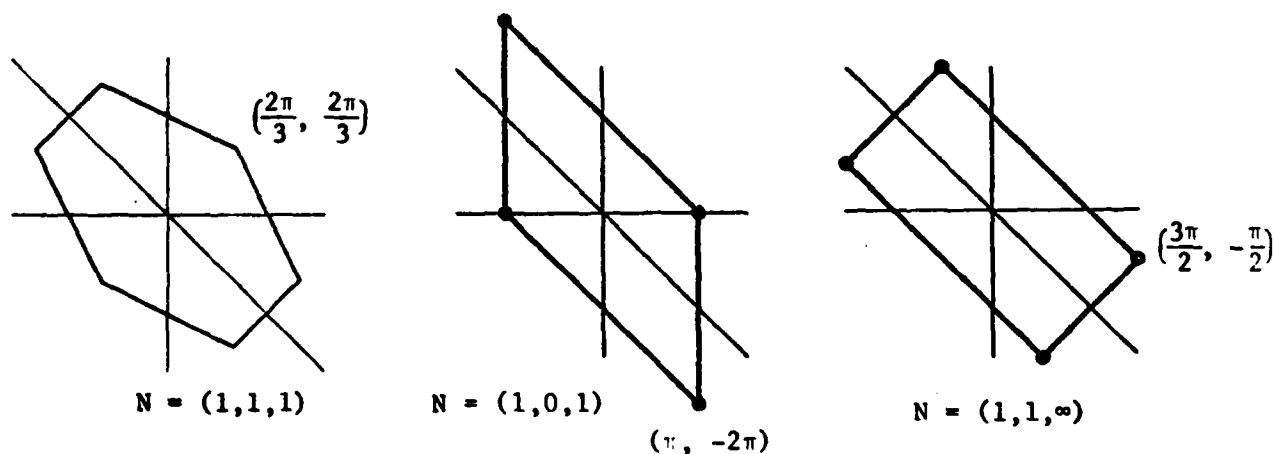


Figure 2

A detailed discussion of the properties of the sets Ω_N is given in [3]. We merely note that they are fundamental domains, i.e., up to a set of measure zero, their translates $2\pi j + \Omega_N$, $j \in \mathbb{Z}^2$, form a partition of \mathbb{R}^2 .

Our first result is an extension of Theorem 5.2 in [3] to include interpolation of data with power growth as was done in [7] for the univariate case.

Theorem 1. Assume that the Fourier transform of f is a tempered distribution with $\text{supp } \hat{f} \subset \Omega_N$. If the sequence n satisfies (n1) - (n3), then, for any $\alpha \in \mathbb{Z}_+^2$, the partial derivative $D^\alpha I_n f$ of the cardinal interpolant converges locally uniformly on \mathbb{R}^2 to $D^\alpha f$, as $n \rightarrow \infty$.

As for the univariate case, the converse of the above Theorem holds with " $\subset \Omega_N$ " replaced by " $\subseteq \overline{\Omega_N}$ ":

Theorem 2. Assume that the sequence n satisfies (n1) - (n3). If a sequence of cardinal splines $s_n \in S_n$ converges locally uniformly to f and if $|s_n(x)| \leq c(1 + |x|)^c$ for all n and some $c > 0$, then $\text{supp } \hat{f} \subseteq \overline{\Omega_N}$.

We may relax the assumption (n2). Clearly any subsequence of n also satisfies (n1) and (n3). If $\{N_\alpha\}$ are the limit points of the sequence n/n' then one has to replace the set Ω_N in the Theorems by $\bigcap_\alpha \Omega_{N_\alpha}$. The figure below shows the intersection Ω and the union Ω_U of all possible limit sets.

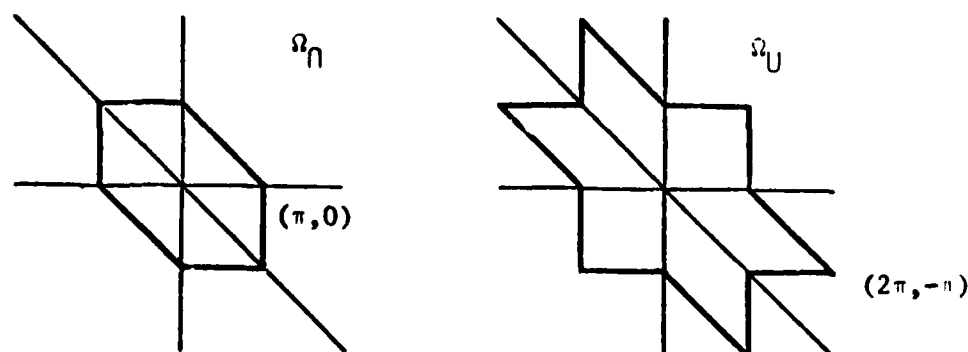


Figure 3

2. Proofs. We assume throughout that the sequence n satisfies (n1) - (n3). By c we denote various positive generic constants which do not depend on n . These constants may change even within the same line. Further, we set

$$(1) \quad d_n(x) := \text{dist}(x, \partial\Omega_n)$$

and denote by χ_n the characteristic function of the set Ω_n .

Denote by $L_n \in S_n$ the fundamental spline which interpolates the data $\delta_{0,k}$, $k \in \mathbb{Z}^2$. It is easily seen [3] that L_n decays exponentially at infinity. Therefore, if we, e.g., assume that

$$(2) \quad |f(x)| \leq c(1 + |x|)^c,$$

then we can write the cardinal interpolant in Lagrange form

$$(3) \quad I_n f = \sum_{j \in \mathbb{Z}^2} f(j) L_n(\cdot - j).$$

The proof of Theorems 1 and 2 is based on the following estimate for the Fourier transform of L_n which will be derived at the end of this section.

Theorem 3. For any $\varepsilon > 0$ and $\alpha \in \mathbb{Z}_+^2$ there exists n'_0 such that for $n' \geq n'_0$ and $d(x) > \varepsilon$

$$(4) \quad |D^\alpha(\hat{L}_n(x) - \chi_N(x))| \leq (1 + cd_N(x))^{-n'}.$$

Proof of Theorem 1. Denote by S [4] the space of rapidly decreasing test functions $\phi \in \mathbb{R}^2$. The assumption $\hat{f} \in S'$ and $\text{supp } \hat{f} \subset \Omega_N$ implies (2) and hence the representation (3) is valid for the cardinal interpolant.

Set

$$\hat{f}_K := \sum_{|j| < K} f(j) e^{-ij^2}.$$

Since Ω_N is a fundamental domain which contains $\text{supp } \hat{f}$, the values

$$f(-j) = (2\pi)^{-2} \langle \hat{f}, e^{ij^2} \rangle$$

are the Fourier coefficients of \hat{f} . Therefore \hat{f}_K converges in S' to the periodic extension of \hat{f} :

$$\hat{f}^\circ := \sum_{j \in \mathbb{Z}} \hat{f}(\cdot + 2\pi j).$$

This means that there exists $\gamma \in \mathbb{Z}_+$ such that for any $\psi \in S$,

$$(5) \quad |\langle \hat{f}^\circ - \hat{f}_K, \psi \rangle| = o(1) \cdot \|\psi\|_\gamma, \text{ as } K \rightarrow \infty,$$

where

$$\|\psi\|_\gamma := \max_{\alpha, \beta \leq \gamma} \sup_{y \in \mathbb{R}} |y^\alpha D^\beta \psi(y)|.$$

Note that (5) implies

$$(5') \quad |\langle \hat{f}_K, \psi \rangle| \leq c \|\psi\|_\gamma,$$

uniformly in K . Putting $\phi(y) := (iy)^\alpha e^{ixy}$ we can write the difference

$$(2\pi)^2 D^\alpha (f - I_n f)(x)$$

in the form

$$\langle \hat{f}, \phi \rangle - \lim_{K \rightarrow \infty} \langle \hat{f}_K, \hat{L}_n \phi \rangle =$$

$$\langle \hat{f}, (1 - \hat{L}_n) \phi \rangle - \lim_{K \rightarrow \infty} \langle \hat{f}_K - \hat{f}, \hat{L}_n \phi \rangle.$$

Since $\text{supp } \hat{f} \subset \Omega_N$ the first term can be estimated using Theorem 3

with $\varepsilon := \text{dist}(\text{supp } \hat{f}, \partial\Omega_N)/2$. For the second term, we choose a cut-off function $w \in S$ with $\text{supp } w \subseteq \Omega_\varepsilon := (\Omega_N \cup \{y: d_N(y) < \varepsilon\})$ and with $w(y) = 1$ for $y \in \Omega_{\varepsilon/2}$. Since $((2\pi j) + \text{supp } \hat{f}) \cap \text{supp } w = \emptyset$ for $j \neq 0$ and $\text{supp } \hat{f} \cap \text{supp}(1-w) = \emptyset$ we have

$$\langle \hat{f} - \hat{f}_K, \hat{L}_n \phi \rangle = \langle \hat{f} - \hat{f}_K, w \hat{L}_n \phi \rangle + \langle -\hat{f}_K, (1-w) \hat{L}_n \phi \rangle.$$

The first term tends to zero as $K \rightarrow \infty$. As to the second term note that $\text{dist}(\text{supp}(1-w), \Omega_N) \geq \varepsilon/2$, which by (4) implies

$$\|(1-w) \hat{L}_n \phi\|_Y \rightarrow 0, \quad n \rightarrow \infty.$$

It follows that

$$|\langle \hat{f}, \phi \rangle - \lim_{K \rightarrow \infty} \langle \hat{f}_K, \hat{L}_n \phi \rangle|$$

tends to zero, uniformly for bounded x (cf. definition of ϕ).

Proof of Theorem 2. Let $\phi \in S$ and assume that $\text{supp } \phi \cap \overline{\Omega_N} = \emptyset$. If the sequence $s_n \in S_n$ converges locally uniformly to f then (2) holds and we have

$$\langle f, \hat{\phi} \rangle = \lim_{n \rightarrow \infty} \left(\lim_{K \rightarrow \infty} \sum_{|j| \leq K} \langle s_n(j) L_n(\cdot - j), \hat{\phi} \rangle \right).$$

Let $\Delta := \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$ denote the Laplace operator. Since \hat{L}_n together with all its derivatives vanishes at infinity, we have

$$\begin{aligned} \langle s_n(j) L_n(\cdot - j), \hat{\phi} \rangle &= \langle s_n(j) e^{-1j \cdot} \hat{L}_n, \phi \rangle \\ &= \langle (1 + |j|^2)^{-2-c/2} s_n(j) e^{-1j \cdot}, (1-\Delta)^{2+c/2} (\hat{L}_n \phi) \rangle. \end{aligned}$$

Applying Theorem 3 with $\varepsilon = \text{dist}(\text{supp } \phi, \overline{\Omega}_N) > 0$ we have for sufficiently large n' ,

$$|\langle \dots \rangle| \leq c(1 + |j|^2)^{-2} \left[\sup_j (1 + |j|^2)^{c/2} |s_n(j)| \right] (1 + c\varepsilon)^{-n'} \|\phi\|_Y.$$

It follows that $\langle \hat{f}, \phi \rangle = 0$.

For the proof of Theorem 3 we make use of the following precise estimates for \hat{L}_n and the numbers $a_{n,j}$ which have been derived in [3].

Theorem 4 [3, Thm. 5.2].

$$(6) \quad |\hat{L}_n(x) - \chi_n(x)| \leq c(1 + c d_n(x))^{-n'}.$$

Proposition 1. [3, Prop. 5.2, Lemmas 6.5, 6.6]. Set $J' := \{\pm(1,1), \pm(2,-1), \pm(-1,2)\}$. Then, for $(2\pi x) \in \Omega_n$, we have

$$(7) \quad |a_{n,j}(x)| \leq \begin{cases} (1 + c \text{dist}(2\pi x, \Gamma_{n,j} \cap \Omega_n))^{-n'}, & j \in J \\ (1 + c \text{dist}(2\pi x, -j/2))^{-n'}, & j \in J' \\ (1 + c|j|)^{-n'}, & j \in \mathbb{Z}^2 \setminus \{J \cup J' \cup 0\}. \end{cases}$$

Proposition 2. [3, Prop. 5.1]. Ω_n depends continuously on n in the Hausdorff topology.

The reader who compares these statements with those in [3] will notice that we have slightly changed the notation. Note that the estimate (6) is stronger than the assertion of Theorem 3 for $\alpha = 0$ since the constants c in (6) do not depend on the distance of x to $\partial\Omega_n$.

We need the analogue of estimate (7) for the derivatives of $a_{n,j}$.

Lemma 1. For any $\delta > 0$, there exist constants c_δ, c and $n_0(\delta)$ such that for all $n' \geq n_0$ and $2\pi x \in \Omega_n$

$$(8) \quad |D^\alpha a_{n,j}(x)| \leq c_\delta (1 + c\delta)^{n'} \begin{cases} [1 + c \operatorname{dist}(2\pi x, \Gamma_{n,j} \cap \Omega_n)]^{-n'}, & j \in J \\ [1 + c \operatorname{dist}(2\pi x, -j/2)]^{-n'}, & j \in J' \\ [1 + c|j|]^{-n'}, & j \in \mathbb{Z}^2 \setminus (J \cup J' \cup 0). \end{cases}$$

The proof of this Lemma is technical and we postpone it until the end.

Proposition 3. Let $x' = x + j$ with $j \in \mathbb{Z}^2 \setminus 0$ and $2\pi x \in \Omega_n$. Then, for any $\delta > 0$, there exist constants c_δ, c and $n_0(\delta)$ such that for all $n' \geq n_0$

$$(9) \quad |D^\alpha a_{n,j}(x)| \leq c_\delta (1 + c\delta)^{n'} (1 + c d_n(2\pi x'))^{-n'}.$$

For $\alpha = 0$ this is Proposition 5.4 in [3]. There we bounded the terms in square brackets on the right hand side of (8) by $(1 + c d_n(2\pi x'))$ which appears on the right hand side of (9). Clearly, the case $\alpha \neq 0$ can be treated in the same way.

Proof of Theorem 3. Since

$$\frac{1}{\hat{L}_n(2\pi x)} = \frac{P_n(2\pi x)}{\hat{M}_n(2\pi x)} = \sum_{j \in \mathbb{Z}^2} a_{n,j}(x),$$

we have, for $|\alpha| = 1$,

$$(2\pi) D^\alpha \hat{L}_n(2\pi x) = -\hat{L}_n(2\pi x)^2 \sum_{j \neq 0} D^\alpha a_{n,j}(x).$$

For arbitrary $\alpha \neq 0$ it follows that

$$(10) \quad (2\pi)^{|a|} D^{\alpha} \hat{L}_n(2\pi x) = \sum_{j \neq 0} \sum_{\beta_1 + \beta_2 < \alpha} c_{\beta} D^{\beta_1} \hat{L}_n(2\pi x) D^{\beta_2} \hat{L}_n(2\pi x) D^{\alpha - \beta_1 - \beta_2} a_{n,j}(x).$$

Let us first assume that $2\pi x \in \Omega_n$. We claim that for any $\delta > 0$ there exists c_{δ} such that

$$(11) \quad |D^{\alpha} \hat{L}_n(2\pi x)| \leq c_{\delta} (1 + c\delta)^{n'} (1 + cd_n(2\pi x))^{-n'}, \quad 2\pi x \in \Omega_n.$$

For $\alpha = 0$ this is a weaker statement than the assertion of Theorem 4. Using induction on $|a|$ it is sufficient to show that

$$\sum_{j \neq 0} D^{\alpha} a_{n,j}(x)$$

can be bounded by the right hand side of (11). Lemma 1 yields

$$|D^{\alpha} a_{n,j}(x)| \leq c_{\delta} (1 + c\delta)^{n'} \cdot \begin{cases} (1 + cd_n(2\pi x))^{-n'}, & j \in J \cup J', \\ (1 + c|j|)^{-n'}, & j \in \mathbb{Z}^2 \setminus \{J \cup J' \cup 0\}. \end{cases}$$

Summing this inequality over $j \in \mathbb{Z}^2 \setminus 0$ finishes the proof of (11).

Secondly, let $x' = x + j$ with $(2\pi x) \in \Omega_n$. Then, writing

$$\hat{L}_n(2\pi x') = \hat{L}_n(2\pi x) a_{n,j}(x),$$

we see that

$$D^{\alpha} \hat{L}_n(2\pi x') = \sum_{\beta \leq \alpha} c_{\beta} D^{\beta} \hat{L}_n(2\pi x) D^{\alpha - \beta} a_{n,j}(x).$$

Therefore, by (11) and Proposition 3, $D^{\alpha} \hat{L}_n$ can be estimated by

$$(12) \quad |D^{\alpha} \hat{L}_n(2\pi x')| \leq c_{\delta} (1 + c\delta)^{n'} (1 + cd_n(2\pi x'))^{-n'},$$

$$x' = x + j, \quad 2\pi x \in \Omega_n.$$

Theorem 3 easily follows from the estimates (11) and (12): Let $\epsilon > 0$ and assume that $d_N(2\pi x) > \epsilon$. We choose n'_0 so that $d_n(2\pi x) > d_N(2\pi x)/2$ for $n' > n'_0$. Now (11) and (12) give (4) since we can choose δ sufficiently small.

Proof of Lemma 1. In proving (8) we make use of the symmetries of the mesh. If A is a linear transformation which leaves the set J invariant, we have

$$a_{n,Aj}(Ax) = a_{\bar{n},j}(x)$$

where \bar{n} is the appropriate permutation of n . Similarly,

$$A\Omega_n = \Omega_{\bar{n}}.$$

From this one can check (cf. [3, section 3]) that one may assume, by changing n if necessary, that $x := (u, v)$ lies in the first quadrant. Further, since the roles of u and v may be interchanged and $(2\pi x) \in \Omega_n \subset \Omega$, (cf. Figure 3) we shall assume throughout this proof that

$$(13) \quad 0 \leq v \leq u \leq 1/2.$$

By definition (1.5) of $a_{n,j}$ we have

$$(14) \quad |D^{\alpha} a_{n,j}(u,v)| \leq c|n|^{\alpha} \sum \left[\frac{|u|^{n_1-\beta_1}}{|u+k|^{n_1+\beta_4}} \frac{|v|^{n_2-\beta_2}}{|v+l|^{n_2+\beta_5}} \frac{|u+v|^{n_3-\beta_3}}{|u+v+k+l|^{n_3+\beta_6}} \right]$$

where the sum is taken over all β which satisfy

$$(15) \quad \begin{aligned} 0 \leq \beta_v \leq n_v, \quad v = 1, 2, 3, \\ \sum_{v=1}^6 \beta_v = \alpha_1 + \alpha_2 \end{aligned}$$

$$\beta_v = -\beta_{3+v} \text{ in case } \begin{cases} k = 0 & v = 1 \\ l = 0 & v = 2 \\ k+l = 0 & v = 3. \end{cases}$$

This last restriction comes from the fact that, e.g., for $k = 0$, the factor $\left(\frac{u}{u+k}\right)^{n_1}$ is equal to 1, hence does not figure in the differentiation. To estimate the individual summands in (14) we consider 4 cases. Unless $(k, l) \in \{(0, -1), (-1, 1)\}$ (cases (ii)(b), (c)) we bound each summand [...] on the right hand side of (14) by

$$(16) \quad c_{\delta}(1 + c\delta)^{n'} \max((1 + c|j|)^{-n'}, |a_{n,j}(x)|).$$

$$(1) \quad \underline{v} \leq \delta < 1/4, \quad u \leq 1/8:$$

Using the inequality

$$(17) \quad \left| \frac{p}{p+q} \right| \leq (1 + c_{\epsilon}|q|)^{-1}, \quad q \in \mathbb{Z} \setminus 0, \quad |p| < 1/2 - \epsilon,$$

we obtain the estimate

$$|[\dots]| \leq c(1 + c|k|)^{-n_1 + \beta_1} (1 + c|l|)^{-n_2 + \beta_2} (1 + c|k+l|)^{-n_3 + \beta_3}$$

with c involving terms like $|u+k|^{-(\beta_1 + \beta_4)}$, $k \neq 0$, which are bounded.

Since at most one of the components of n is less than n' , this implies (8).

$$(11) \quad \underline{v \leq \delta < 1/4, \quad u > 1/8} :$$

We consider several subcases.

$$(a) \quad (k, l) \notin \{(0, 1), (-1, 1)\}:$$

We have

$$(18) \quad |[\dots]| \leq c(1 + c|l|)^{-n_2 + \beta_2} \left| \frac{u}{u+k} \right|^{n_1} \left| \frac{u+v}{u+v+k+l} \right|^{n_3}.$$

This can be estimated as before unless $k = -1$ or $k+l = -1$. If $k = -1$ and $l \neq 0, 1$ we can write the right side of (18) as

$$c(1 + c|l|)^{-n_2 + \beta_2} |a_{n, (-1, 0)}(u, v)| \left| \frac{u+v-1}{u+v-1+l} \right|^{n_3}.$$

The last factor is less than $(1 + c|l|)^{-n_3}$ and, since for $2\pi(u, v) \in \Omega_n$, $|a_{n, (-1, 0)}(u, v)| < 1$, (8) follows. If $(k, l) = (-1, 0)$ it is easily seen that the left hand side of (18) can be bounded by $c|a_{n, (-1, 0)}(u, v)|$. The cases $k + l = -1$, $k \neq 0, -1$ are treated similarly.

$$(b) \quad (k, l) = (0, -1):$$

Set $n_\beta := (n_1, n_2 - \beta_2, n_3)$. By Proposition 2, there exists

$n_0 = n_0(\alpha, \delta)$ so that the boundaries of Ω_n and Ω_{n_β} are within δ of the boundary of the limit set Ω_N for $n' \geq n_0$ and all β_2 satisfying

(15). Moreover, $\frac{1}{2} n' \leq n'_\beta \leq 2n'$, $n' \geq n_0$.

For $u+v \leq 1/2$ (and $n' \geq n_0$) we obtain, using also Proposition 1,

$$\begin{aligned} |[\dots]| &\leq c |a_{n_\beta, (0, -1)}(u, v)| \\ &\leq c(1 + c \operatorname{dist}(2\pi x, \Gamma_{n_\beta, (0, -1)} \cap \Omega_{n_\beta}))^{-n'_\beta} \\ &\leq c(1 + c\delta)^{n'} (1 + c \operatorname{dist}(2\pi x, \Gamma_{n, (0, -1)} \cap \Omega_n))^{-n'}. \end{aligned}$$

For the last inequality we have used that

$$\operatorname{dist}(\Gamma_{n_\beta, (0, -1)} \cap \Omega_{n_\beta}, \Gamma_{n, (0, -1)} \cap \Omega_n) \leq \delta.$$

For $1/2 < u+v \leq 1/2 + \delta$ we have

$$\begin{aligned} |[\dots]| &\leq c \left| \frac{v}{1-v} \right|^{n_2 - \beta_2} \left| \frac{u+v}{1-u-v} \right|^{n_3} \\ &\leq c \left| \frac{v}{1-v} \right|^{n_2 - \beta_2} \min \left\{ \left| \frac{u+v}{1-u-v} \right|^{n_3}, \left| \frac{1-u}{u} \right|^{n_1} \right\} \end{aligned}$$

where we have used that $a_{n, (-1, 0)}(u, v) < 1$. By our assumptions on u and v the minimum can be estimated by $(1 + c\delta)^{n'}$. Therefore, if $n_2 \geq cn'$, (8) follows. If $\lim_{n \rightarrow \infty} n_2/n' = 0$, the curve $\Gamma_{n, (0, -1)} \cap \bar{\Omega}_n$ converges to the segment $\{2\pi(u, v): u+v = 1/2, (u, v) > (0, 0)\}$. Therefore, we may assume that

$$\operatorname{dist}(2\pi x, \Gamma_{n, (0, -1)} \cap \Omega_n) \leq c\delta$$

for $n' \geq n_0$ and (8) follows.

(c) $(k, l) = (-1, 1)$:

We have

$$\begin{aligned}
|[\dots]| &\leq c \left| \frac{u}{1-u} \right|^{n_1} \left| \frac{v}{1+v} \right|^{n_2 - \beta_2} \\
&\leq c(1+c)^{n_2 - \beta_2} (1 + c(1/2 - u))^{-n_1}.
\end{aligned}$$

Since $\Gamma_{n,(-1,1)}$ does not intersect the square $[0, \pi] \times [-\pi, 0]$, this implies (8).

(iii) $\delta \leq v$, $u < 1/2 - \delta$: Since $v \leq u$, we have

$$|[\dots]| \leq c_\delta \left| \frac{u}{u+k} \right|^{n_1} \left| \frac{v}{v+l} \right|^{n_2} \left| \frac{u+v}{u+v+k+l} \right|^{n_3} \leq c_\delta |a_{n,j}(x)|.$$

(iv) $\delta \leq v$, $u \geq 1/2 - \delta$: We have

$$\begin{aligned}
|[\dots]| &\leq c_\delta |a_{n,j}(x)| \left| \frac{u+v}{u+v+k+l} \right|^{\beta_6} \\
&\leq c_\delta |a_{n,j}(x)| \left| \frac{u+v}{1-u-v} \right|^{\beta_6}.
\end{aligned}$$

Assume e.g., that $n_1 = \min(n_1, n_2)$. Since $2\pi x \in \Omega_n$,

$$|a_{n,(-1,0)}(x)| = \left(\frac{u}{1-u} \right)^{n_1} \left(\frac{u+v}{1-u-v} \right)^{n_3} < 1.$$

From this and the fact that $u \geq 1/2 - \delta$, $\beta_6 \leq n_3$, we have

$$\left| \frac{u+v}{1-u-v} \right|^{\beta_6} \leq \left| \frac{1-u}{u} \right|^{\frac{n_1}{n_3} \beta_6} \leq (1 + c\delta)^{n'}.$$

Combining the above estimates yields

$$|[\dots]| \leq c_\delta |a_{n,j}(x)| (1 + c\delta)^{n'}.$$

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